JOURNAL OF COMBINATORIAL THEORY, Series A 40, 1–8 (1985)

A Symmetric BIBD with Trivial Automorphism Group

JOHN WESLEY BROWN AND E. T. PARKER

University of Illinois, Urbana, Illinois 61801

Communicated by Marshall Hall, Jr.

Received April 4, 1978

Hall and Connor (*Canad. J. Math.* **6** (1954), 35–41) give iff conditions for a residual design to be embeddable in a symmetrical design. This article gives the corresponding theorem and proof for derived designs and uses this theorem to complete a BIBD (15, 35, 14, 6, 5) (Hall, "Combinatorial Theory," Ginn (Blaisdell), Boston, p. 295, #83; Rao, (Sankhyā **23** (1961), 117–127), #62) to 13 non-isomorphic SBIBD(36, 15, 6)s. Several of these symmetric designs have the trivial automorphism group, and one of them provides an example of a SBIBD(36, 21, 12) with a symmetric incidence matrix that has all 0s on the diagonal. © 1985 Academic Press, Inc.

Given a SBIBD(v, k, λ) we can always obtain two designs $(v', b', r', k', \lambda')$ from it. The *derived* design BIBD$(k, v-1, k-1, \lambda, \lambda-1)$, by deleting a given block and deleting all occurrences of objects not in that block from the other blocks, and the *residual* design BIBD$(v-k, v-1, k, k-\lambda, \lambda)$, by deleting a given block and deleting all occurrences of objects in that block from the other blocks [4, p. 103].

1. General Conditions for Embedding

For residual designs we have the

THEOREM (Hall and Connor). *A design D with parameters satisfying $r=k+\lambda$, $v\lambda=k(k+\lambda-1)$, $b\lambda=(k+\lambda)(k+\lambda-1)$ can be embedded as a residual design in a symmetric design $S\equiv$ SBIBD$(b+1, r, \lambda)=$ SBIBD$(v+k+\lambda, b+1, r, k+\lambda, \lambda)$ iff we can find in D sets of blocks S_j, $j=1,\ldots, k+\lambda\ni$:*

(1) *Each S_j consists of $k+\lambda-1$ blocks of D.*

(2) *The blocks of an S_j together contain each element of D λ times.*

(3) *Any two distinct sets S_i, S_j have exactly $\lambda-1$ blocks in common.*

(4) *Any blocks of D is in precisely λ sets S_j.*

0097-3165/85 $3.00

Furthermore, if two blocks B_m, B_n in D have l elements in common they must occur together in exactly $\lambda - l \geqslant 0$ sets S_j.

Inspection of the derived design parameters yields $v' = r' + 1$, $k' = \lambda' + 1$, $v'r' = b'(\lambda' + 1)$. The corresponding theorem for derived designs is

THEOREM. *A design D with parameters satisfying*

$$v = r + 1, \qquad k = \lambda + 1, \qquad vr = b(\lambda + 1)$$

can be embedded as a derived design in a symmetric design $S \equiv$ SBIBD$(b+1,\ r+1,\ \lambda+1)$ iff we can find in D sets of blocks S_j, $j = 1,\dots, b - v + 1 \ni$:

(1) *Each S_j consists of $b - v$ blocks of D.*

(2) *The blocks of S_j together contain each element $r - k$ times.*

(3) *Any two distinct sets S_i, S_j have exactly $b - 2v + k$ blocks in common.*

(4) *Any block of D is in precisely $b - 2v + k + 1$ sets S_j.*

Furthermore, if two blocks B_m, B_n in D have l elements in common, they must occur together in exactly $(b - 2v + k + 1) - v + 2k - l$ sets S_j.

Proof. Let us adjoin to D new elements $x_1,\dots, x_{b-v+1}$ and a new block B_0 consisting of the v original elements. We adjoin the elements x_j to all blocks not in S_j and to no other blocks. Then the new array contains $b + 1$ blocks and $v + b - v + 1 = b + 1$ elements. The block B_0 contains $v = r + 1$ elements, and by (4) we have adjoined exactly $(b - v + 1) - (b - 2v + k + 1) = v - k$ new elements to each old block. Therefore, in S, each block contains $v - k + k = r + 1$ elements. Each old element appears in $r + 1$ blocks of S, and by (1) each new element appears in $b - (b - v) = r + 1$ blocks of S. Each pair of distinct elements d_i, d_j of D occur together in D λ times and once more in B_0, giving $\lambda + 1$ altogether. By (2) each x_j occurs together with d_i $r - (r - k) = k = \lambda + 1$ times. By (3) any two distinct x_i, x_j occur together in $b - (b - 2v + k) - 2(b - v - (b - 2v + k)) = k = \lambda + 1$ blocks. Finally, if B_m, B_n have l elements in common, there are $\lambda + 1 - l$ S_j which contain neither. Therefore, there are $(b - v + 1) - (\lambda + 1 - l) - 2(v - 2k + l) = (b - 2v + k + 1) - v + 2k - l$ which contain both. Conversely, if we drop from a symmetric design S a block B_0 and all of the elements not in B_0 from the other blocks, it is easily verified that the blocks of S which do not contain a suppressed element x_j form in the derived design D a set of blocks S_j, and the sets S_j have all of the above properties.

2. Application to SBIBD(36, 15, 6)

We start with the BIBD(15, 35, 14, 6, 5) given in the first six positions of blocks 1–35, Fig. 3 [Rao [7, No. 62], Hall [4, No. 82]. According to the embedding theorem we need to find S_j, $j=1,21 \ni$ each S_j contains 20 blocks. Each element occurs 8 times in the blocks of each S_j, any two S_i, S_j have 11 blocks in common, and every block must be in 12 S_j. Furthermore we have the table:

Number of elements in common	4	3	2	1
Number of S_j which contain both blocks	5	6	7	8

Let $D(i)$, $i=1, 35$, be the 6-tuple given by the first 6 positions of block i, Fig. 3. First, we analyse the structure of a single S_j. Let $x_i=1$ if $D(i) \in S_j$, and $x_i=0$ otherwise. By (2) we obtain 15 equations:

$$\text{Eq. } \infty: \quad x_1+x_2+x_3+x_4+x_5+x_6+x_7+x_8+x_9+x_{10}+x_{11} + x_{12}+x_{13}+x_{14}=8$$

$$\vdots$$

$$\text{Eq. } 9: \quad x_1+x_2+x_3+x_6+x_{10}+x_{16}+x_{17}+x_{21}+x_{23}+x_{24} + x_{28}+x_{30}+x_{31}+x_{35}=8$$

$$\vdots$$

$$\text{Eq. } 13: \quad x_3+x_5+x_6+x_7+x_{14}+x_{18}+x_{20}+x_{21}+x_{25}+x_{27} + x_{28}+x_{32}+x_{34}+x_{35}=8$$

and from (1) we obtain Eq. 16: $\sum_{i=1}^{35} x_i=20$. If we subtract Eq.∞ from $\sum_{k=0}^{6}$ Eq. k, we obtain $3\sum_{i=8}^{35} x_i=48$ or $\sum_{i=8}^{35} x_i=16$. Subtracting Eq. 16 from this gives $\sum_{i=8}^{14} x_i=4$, and subtraction of this from Eq.∞ gives $\sum_{i=1}^{7} x_i=4$.

Let $P=(0\ 1\ 2\ 3\ 4\ 5\ 6)(7\ 8\ 9\ 10\ 11\ 12\ 13)$. Then P to any power is an automorphism of the design. Therefore, we can restrict the search to the 5 cases $x_1=x_2=x_3=x_4=1$; $x_1=x_2=x_3=x_5=1$; $x_1=x_2=x_3=x_6=1$; $x_1=x_2=x_4=x_5=1$; $x_1=x_2=x_4=x_6$. All other solutions will differ from these by a power of P.

Now consider $\{D(15), D(28), D(33)\}$. Each block in this set has 4 elements in common with each of the other two. Therefore there are 5 exactly S_j which contain any two of them. Let y_m, $m=0, 1, 2, 3$, be the number of sets S_j which contain m of the three blocks. Then we have the equations:

$$y_0+y_1+y_2+y_3=21$$

$$y_2+3y_3=5\cdot 3=15$$

$$y_1+2y_2+3y_3=12\cdot 3=36.$$

(11100101100101101110010000100111111)
(11100101001011110111001000101011101)
(11100101100101001010011010111011110)
(11100101100101111010000010101011111)
(11100101100101101110111010100010110)
(11100100101110000111111010001010111)
(11100101100101100010011110100011111)
(11100101100101111011010110100001101)
(11100101100101101011111110000001110)
(11100101001011100100001001101111111)
(11100101001011001101011001111101100)
(11100101001011111101000001101101101)
(11100100101110001111010001011101011)
(11100101001011111111001001111001000)
(11100101001011101101101001001101110)
(11100101001011100101011101100101101)
(11100100101110000110011011111010011)
(11100100101110011111010011111000001)
(11100100101110001101110011001100111)
(11100101001011111101101011101000100)
(11100100101110001111111011011000010)
(11100100101110111111100011001000011)
(11100101100101101000010111100101111)
(11100101100101101010011111110001010)
(11100101001011110001001111101001101)
(11100100101110000011011111011001011)
(11100100101110100111111111000000011)

FIG. 1. 27 Solutions to the constraints on the x_i.

A. (11100101100101101110010000100111111)
B. (11100101001011111101101011101000100)
C. (11100100101110000011011111011001011)

D. (11001011001011000100111101010111110)
E. (11001010010111111010100011001011011)
F. (11001011011100011111010110110000101)

G. (10010110010111001001011010111111100)
H. (10010110111001111110001101110001010)
I. (10010110101110110101100110000110111)

J. (00101111011100101011101100001101110)
K. (00101111111001011000001111110001110 1)
L. (00101110101110011110110001111110000)

M. (01011100111001010111111000001011101)
N. (01011101011100100100001011111110011)
P. (01011101100101111001110111010101000)

Q. (10111001110010101111111010010010010)
R. (10111001001011110011000100111111001)
S. (10111000111001001000110111101100111)

T. (01110010010111100111111101100100001)
U. (01110011100101010010101110111010110)
V. (01110011110010011101000001011101111)

FIG. 2. 21 Compatible S_j.

This system has the unique solution in non-negative integers (0, 6, 15, 0) and this gives us the inequalities $1 \leqslant x_{15} + x_{28} + x_{33} \leqslant 2$. The same analysis applies to all triples $\{P^i(D(15)),\ P^i(D(28)),\ P^i(D(33))\}$ and $\{P^i(D(15)),\ P^i(D(22)),\ P^i(D(29))\}$, $i = 0, \ldots, 6$, giving 14 inequality couples in all. The system of linear diophantine constrains on the x_i was found to have 27 solutions (Fig. 1) for the case $x_1 = x_2 = x_3 = x_6$ and none for the other four cases. Let X designate this solution set.

Now consider the set $A = \{D(1), D(2), D(3), D(6), D(10)\}$, each of these blocks has 3 elements in common with each of the other 4 blocks. Let

0	(∞	0	1	2	3	4	5	6	7	8	9	10	11	12	13)
1	(∞	0	7	8	9	11	J	K	L	M	N	P	T	U	V)
2	(∞	1	8	9	10	12	G	H	I	J	K	L	Q	R	S)
3	(∞	2	9	10	11	13	D	E	F	G	H	I	M	N	P)
4	(∞	3	10	11	12	7	A	B	C	D	E	F	J	K	L)
5	(∞	4	11	12	13	8	A	B	C	G	H	I	T	U	V)
6	(∞	5	12	13	7	9	D	E	F	Q	R	S	T	U	V)
7	(∞	6	13	7	8	10	A	B	C	M	N	P	Q	R	S)
8	(∞	7	0	6	5	3	C	E	G	H	I	L	M	S	T)
9	(∞	8	1	0	6	4	B	D	E	F	G	J	N	R	T)
10	(∞	9	2	1	0	5	A	B	C	D	I	L	P	R	U)
11	(∞	10	3	2	1	6	A	E	G	K	P	Q	T	U	V)
12	(∞	11	4	3	2	0	B	D	H	K	M	Q	R	S	V)
13	(∞	12	5	4	3	1	A	F	H	J	M	N	P	S	U)
14	(∞	13	6	5	4	2	C	F	I	J	K	L	N	Q	V)
15	(1	2	4	7	8	10	C	D	F	G	L	M	S	U	V)
16	(2	3	5	8	9	11	A	C	D	G	J	N	Q	S	T)
17	(3	4	6	9	10	12	C	D	I	K	M	N	R	T	U)
18	(4	5	0	10	11	13	C	E	G	J	K	P	R	S	U)
19	(5	6	1	11	12	7	B	D	G	I	K	N	P	S	V)
20	(6	0	2	12	13	8	A	D	E	H	K	L	N	S	U)
21	(0	1	3	13	7	9	A	C	F	G	H	K	N	R	V)
22	(2	3	5	7	8	10	B	E	H	I	J	N	R	U	V)
23	(3	4	6	8	9	11	A	E	F	I	L	P	R	S	V)
24	(4	5	0	9	10	12	A	B	E	G	L	M	N	Q	V)
25	(5	6	1	10	11	13	A	D	H	J	L	M	R	T	V)
26	(6	0	2	11	12	7	A	F	G	I	J	M	Q	R	U)
27	(0	1	3	12	13	8	C	D	E	I	J	M	P	Q	V)
28	(1	2	4	13	7	9	A	B	E	I	J	K	M	S	T)
29	(0	4	5	7	8	10	A	D	F	H	I	K	P	Q	T)
30	(1	5	6	8	9	11	B	C	E	F	H	K	M	Q	U)
31	(2	6	0	9	10	12	B	C	F	H	J	P	S	T	V)
32	(3	0	1	10	11	13	B	F	I	L	N	Q	S	T	U)
33	(4	1	2	11	12	7	C	E	H	L	N	P	Q	R	T)
34	(5	2	3	12	13	8	B	F	G	K	L	M	P	R	T)
35	(6	3	4	13	7	9	B	D	G	H	J	L	P	Q	U)

FIG. 3. SBIBD(36, 15, 6): Derived (15, 36, 14, 6, 5); Residual (21, 35, 15, 9, 6).

z_n, $n=0,\ldots,5$, be the number of S_j which contain n of these blocks. Then we have the equations

$$z_0+z_1+z_2+z_3+z_4+z_5=21$$

$$z_2+3z_3+6z_4+10z_5=10\cdot 6=60$$

$$z_1+2z_2+3z_3+4z_4+5z_5=5\cdot 12=60.$$

Now since $1\leqslant x_{16}+x_{23}+x_{30}\leqslant 2$ and $1\leqslant x_{21}+x_{28}+x_{35}\leqslant 2$, $z_0=z_1=0$. The system is then seen to have two solutions: (0, 0, 6, 12, 3, 0) and (0, 0, 5, 15, 0, 1). Inspection of the solutions in X shows that $z_5=1$ does not

1	(1	8	9	10	12	18	20	21	24	26	27	29	31	32	36)
2	(2	9	10	11	13	15	19	21	25	27	28	30	32	33	36)
3	(3	10	11	12	14	15	16	20	22	26	28	31	33	34	36)
4	(4	8	11	12	13	16	17	21	22	23	27	32	34	35	36)
5	(5	9	12	13	14	15	17	18	23	24	28	29	33	35	36)
6	(6	8	10	13	14	16	18	19	22	24	25	29	30	34	36)
7	(7	8	9	11	14	17	19	20	23	25	26	30	31	35	36)
8	(1	4	6	7	8	15	19	21	22	26	28	29	33	35	36)
9	(1	2	5	7	9	15	16	20	22	23	27	29	30	34	36)
10	(1	2	3	6	10	16	17	21	23	24	28	30	31	35	36)
11	(2	3	4	7	11	15	17	18	22	24	25	29	31	32	36)
12	(1	3	4	5	12	16	18	19	23	25	26	30	32	33	36)
13	(2	4	5	6	13	17	19	20	24	26	27	31	33	34	36)
14	(3	5	6	7	14	18	20	21	25	27	28	32	34	35	36)
15	(2	3	5	8	9	11	16	18	19	22	24	26	27	28	35)
16	(3	4	6	9	10	12	15	16	17	19	20	25	27	29	35)
17	(4	5	7	10	11	13	16	20	21	23	24	25	26	28	29)
18	(1	5	6	11	12	14	15	19	21	22	23	24	25	27	31)
19	(2	6	7	8	12	13	15	16	18	19	20	23	28	31	32)
20	(1	3	7	9	13	14	16	17	19	20	21	22	24	32	33)
21	(1	2	4	8	10	14	17	18	20	22	23	25	27	28	33)
22	(3	4	6	8	9	11	15	18	20	21	23	24	30	33	34)
23	(4	5	7	9	10	12	17	18	19	21	22	28	30	31	34)
24	(1	5	6	10	11	13	15	17	18	20	22	26	30	32	35)
25	(2	6	7	11	12	14	16	17	18	21	26	27	29	30	33)
26	(1	3	7	8	12	13	15	17	24	25	26	27	28	30	34)
27	(1	2	4	9	13	14	15	16	18	21	25	26	31	34	35)
28	(2	3	5	8	10	14	15	17	19	21	23	26	29	32	34)
29	(1	5	6	8	9	11	16	17	25	28	29	31	32	33	34)
30	(2	6	7	9	10	12	22	23	24	25	26	32	33	34	35)
31	(1	3	7	10	11	13	18	19	23	27	29	31	33	34	35)
32	(1	2	4	11	12	14	19	20	24	28	29	30	32	34	35)
33	(2	3	5	8	12	13	20	21	22	25	29	30	31	33	35)
34	(3	4	6	9	13	14	22	23	26	27	28	29	30	31	32)
35	(4	5	7	8	10	14	15	16	24	27	30	31	32	33	35)
36	(1	2	3	4	5	6	7	8	9	10	11	12	13	14	36)

FIG. 4. Self-dual (36, 15, 6) with trivial automorphism group.

occur, so that every set of 21 S_j must have exactly 3 S_j, each of which has 4 blocks from A. This only occurs with the 4 blocks $D(1)$, $D(2)$, $D(3)$, $D(6)$. Therefore, every set of 21 S_j has exactly 3 solutions from X. Similar argument applies to $P^iA = \{P^i(D(1)), P^i(D(2)), P^i(D(3)), P^i(D(6)), P^i(D(10))\}$, $i = 1, \ldots, 6$, so that we need 3 solutions from each of $P^i(X)$, $i = 1, \ldots, 6$. This gives $7 \cdot 3 = 21$ S_j.

One such set of 7 triples is given in Fig. 2. Each row has 20 1s. Any two distinct rows have 11 1s in common and every column has 12 1s. The condition on the elements is a result of the constraints on the x_i.

The resulting SBIBD(36, 15, 6) is shown in Fig. 3. This design is of interest only as the completion of the example.

```
 1  (1  8  9 10 12 18 20 21 24 26 27 29 31 32 36)
 2  (2  9 10 11 13 15 19 21 25 27 28 30 32 33 36)
 3  (3 10 11 12 14 15 16 20 22 26 28 31 33 34 36)
 4  (4  8 11 12 13 16 17 21 22 23 27 32 34 35 36)
 5  (5  9 12 13 14 15 17 18 23 24 28 29 33 35 36)
 6  (6  8 10 13 14 16 18 19 22 24 25 29 30 34 36)
 7  (7  8  9 11 14 17 19 20 23 25 26 30 31 35 36)
 8  (1  4  6  7  8 15 19 21 22 26 28 29 33 35 36)
 9  (1  2  5  7  9 15 16 20 22 23 27 29 30 34 36)
10  (1  2  3  6 10 16 17 21 23 24 28 30 31 35 36)
11  (2  3  4  7 11 15 17 18 22 24 25 29 31 32 36)
12  (1  3  4  5 12 16 18 19 23 25 26 30 32 33 36)
13  (2  4  5  6 13 17 19 20 24 26 27 31 33 34 36)
14  (3  5  6  7 14 18 20 21 25 27 28 32 34 35 36)
15  (2  3  5  8  9 11 15 16 18 19 21 24 26 34 35)
16  (3  4  6  9 10 12 15 16 17 19 20 25 27 29 35)
17  (4  5  7 10 11 13 16 17 18 20 21 26 28 29 30)
18  (1  5  6 11 12 14 15 17 18 19 21 22 27 30 31)
19  (2  6  7  8 12 13 15 16 18 19 20 23 28 31 32)
20  (1  3  7  9 13 14 16 17 19 20 21 22 24 32 33)
21  (1  2  4  8 10 14 15 17 18 20 21 23 25 33 34)
22  (3  4  6  8  9 11 18 20 22 23 24 27 28 30 33)
23  (4  5  7  9 10 12 19 21 22 23 24 25 28 31 34)
24  (1  5  6 10 11 13 15 20 22 23 24 25 26 32 35)
25  (2  6  7 11 12 14 16 21 23 24 25 26 27 29 33)
26  (1  3  7  8 12 13 15 17 24 25 26 27 28 30 34)
27  (1  2  4  9 13 14 16 18 22 25 26 27 28 31 35)
28  (2  3  5  8 10 14 17 19 22 23 26 27 28 29 32)
29  (1  5  6  8  9 11 16 17 25 28 29 31 32 33 34)
30  (2  6  7  9 10 12 17 18 22 26 30 32 33 34 35)
31  (1  3  7 10 11 13 18 19 23 27 29 31 33 34 35)
32  (1  2  4 11 12 14 19 20 24 28 29 30 32 34 35)
33  (2  3  5  8 12 13 20 21 22 25 29 30 31 33 35)
34  (3  4  6  9 13 14 15 21 23 26 29 30 31 32 34)
35  (4  5  7  8 10 14 15 16 24 27 30 31 32 33 35)
36  (1  2  3  4  5  6  7  8  9 10 11 12 13 14 36)
```

FIG. 5. Symmetric presentation with I ∈ Block I.

In all, four dual pairs and five self-dual SBIBD(36, 15, 6)s can be obtained from the starting design. Several of these have trivial automorphism group. A self-dual design with this property is shown in Fig. 4. Counts on incidences of triples of digits and, dually, triples of blocks are used to determine that no non-trivial automophisms are present.

Finally, the design of Fig. 5 provides a 0, 1 symmetric matrix with all 1s on the diagonal. Thus the complementary SBIBD(36, 21, 12) provides a 0, 1 symmetric matrix with all 0s on the diagonal.

Note. [1 and 2] give many/15417 examples of (36, 15, 6)-designs which are derived from two-graphs with trivial automorphism groups; [3] shows these designs are all distinct, and it may be that some of them also have trivial automorphism group. The example given in this paper was found in 1969. Publication was delayed by a series of oversights by the journal.

References

1. F. C. Bussemaker and J. J. Seidel, Symmetric Hadamard matrices of order 3^6, *in* "Proc. Int. Conf. Combin. Math.," New York Acad. Sci., 1970; TH-Report 70-WSK-02, Technical Univ., Eindhoven.
2. F. C. Bussemaker, R. A. Mathon, and J. J. Seidel, Tables of two-graphs, *in* "Combin. & Graphs" (S. B. Rao, Ed.), Lecture Notes in Math, TH-Report 79-WSK-05, Vol. 5,pp. 70–112, Springer-Verlag, Berlin/New York, 1981; Technical Univ., Eindhoven.
3. W. Haemers, Dual Seidel switching, *in* "Papers dedicated to J. J. Seidel" (P. J. de Doelder, J. de Graaf, and J. H. van Lint, Ed.), pp. 183–191; EUT-Report 84-WSK-03, 1984.
4. M. Hall, Jr., "Combinatorial Theory," Ginn (Blaisdell), Boston, 1967.
5. M. Hall, Jr. and W. S. Connor, An embedding theorem for balanced incomplete block designs, *Canad. J. Math.* **6** (1954), 35–41.
6. E. T. Parker, A result in balanced incomplete block designs, *J. Combin. Theory* **3** (1967), 283–285.
7. C. R. Rao, A study of BIB designs with replications 11 to 15, *Sankhyā* **23** (1961), 117–127.
8. R. G. Stanton, Interconnections of related BIBDS, *J. Combin. Theory* **6** (1969), 387–391.